
Thermodynamic framework for coupling of elasto-viscoplasticity and nonlocal anisotropic damage for microelectronics solder alloys

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Abstract: The microstructure of soldered materials is known to have a strong influence on damage initiation and propagation and being localised. Moreover, it is well-established in the literature that the final failure of a solder joint is preceded by inhomogeneities in the deformation of the joint at relatively early stages, and that predicting the non-uniform micro-damage distribution during thermo-mechanical loading allows one to ultimately predict the failure location and time and then in turn improve the performance and reliability of microelectronic solder alloys. This study develops a general consistent and systematic framework for the analysis of microelectronic solder alloys that assesses a strong coupling between rate-dependent plasticity and rate-dependent damage within the framework of thermodynamic laws and nonlocal gradient-dependent theory. The model presented in this paper can be considered as a feasible thermodynamic approach for microelectronic solder alloys that enables one to derive various coupled thermo-viscoplasticity-viscodamage theories by introducing simplifying assumptions.

Keywords: nonlocal damage; anisotropic damage; viscoplasticity; heterogeneous media; length scale.

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1 Introduction

Solder alloys are extensively used in microelectronics packaging, interconnects, servers, storage and storage array systems, network infrastructure equipment, etc. Generally, solder joints in microelectronics are used for two important functions. Firstly, they form the electrical connection between an electronic component and a printed circuit board substrate, and secondly, they constitute the mechanical bond that holds the component to the substrate. The most commonly used solder alloy up till now is the tin-lead (Sn-Pb) system. However, the global concern about the environmental impact of toxic Pb-based solders in consumer electronics has given an impetus to use Pb-free solder alloys. Some potential Sn-Ag, Sn-Bi, Sn-Zn, Sn-Cu binary eutectic and Sn-Ag-Cu (SAC), Sn-Ag-Bi, Sn-Zn-Bi ternary eutectic solders have been developed as a substitute for Sn-Pb solders (Abtey and Seldavuray, 2000; Matin et al., 2007). Recently, microelectronic industries have shown interest in SAC solder because of its comparatively low melting temperature, the competitive price, and good mechanical properties.

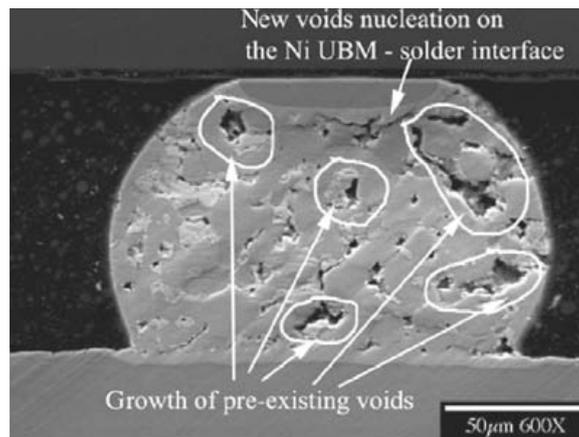
Reliability of soldered joints is an important issue for which an adequate understanding requires knowledge of the material thermo-mechanical behaviour, of loading conditions, and of their evolution during the lifetime of the soldered joint. A solder connection is generally exposed to thermo-mechanical loading during service. The thermo-mechanical loads in solder joints originate from the thermal expansion mismatch between the electronic component and the printed circuit boards and the thermal expansion differences among the different microstructural phases in the solder, as well as from the anisotropy of the Sn matrix (Matin et al., 2005). The induced strain fields due to these loads is expected to be inhomogeneous throughout the microstructure of the solder interconnection, also with inhomogeneities present at different length scales, a heterogeneous nucleation and evolution of microdamages (microcracks and microvoids) will result that ultimately leads to failure of the solder interconnection. Furthermore, solder joints are known to behave differently than the bulk solder material due to the following reasons (Ubachs et al., 2004):

- 1 The presence of a high ratio of substrate surface to solder volume results in a large number of heterogeneous damage nucleation sites.
- 2 A concentration gradient of elemental or metallurgical composition is formed in the joint, e.g. intermetallics.
- 3 Strong size effect such that microstructural sizes are not negligible compared to typical joint dimensions.

Each of these reasons contributes to a structure that is not homogeneous, so that the joint can no longer be regarded as a homogeneous continuum. Therefore, experimental investigations of the microstructure of solder alloys during deformation showed the evolution of localised damage regions as shown in Figure 1 (Ye et al., 2003; 2006). Furthermore, due to ongoing miniaturisation in (micro)electronics, solder alloys are performing at their limit. For example, Sn-Pb solder bumps are already being produced smaller than their minimum size as determined by the surface tension. This leads to a vulnerable connection, which can easily fail when subjected to loads. Also, the increasing functionality and chip density on printed circuit boards, leads to higher energy fluxes and thus higher temperatures. Due to the low homologous temperature, solder materials exhibit creep-fatigue interaction and significant time, temperature, stress and

rate-dependent material characteristics (e.g. Basaran and Chandaroy, 1998; Basaran and Yan, 1998; Conrad et al., 1999; Matin et al., 2005; 2007; Ubachs et al., 2004; 2006; 2007; Gomez and Basaran, 2005; 2007; Geers et al., 2007). Therefore, the main objective of this paper is to formulate thermodynamic-based constitutive equations that combine systematically the aforementioned effects and take into account the proper description of the evolution of localised damage in microelectronic solder alloys.

Figure 1 Scanning electron micrograph of localised damage distribution in flip chip solder joint under current stressing (after Ye et al., 2006)



Several damage models for solder alloys have been proposed based on classical continuum damage mechanics (e.g. Basaran and Chandaroy, 1998; Basaran and Yan, 1998; Stolckarts et al., 1999; 2001; Zhang et al., 2000; Tang and Basaran, 2003; Basaran et al., 2003; Yang et al., 2004; Yang and Nassar, 2005; Wei et al., 2004; 2006; Chen and Chen, 2006; 2007). Generally, these models are formulated using the laws of thermodynamics and continuum damage mechanics concept by Kachanov (1958). However, such classical continuum-based constitutive models are questionable due to the small size of the solder joint in which the microstructural features are comparable in size to the joint itself or with the critical damaged zone. Therefore, a material length scale is required to enter the constitutive equations in order to able to capture the size of localised damage regions independent of the finite element discretisation. On the other hand, development of microstructural-based models for solder joints that take into account the initial microstructure of the joint, the evolution of the microstructure, and the material properties of the individual phases in the joint is computationally very expensive. Moreover, cohesive zone models that consider the interfacial debonding between different phases and external interfaces have been used to predict the reliability and performance of solder alloys (e.g. Erinc et al., 2004; 2007; Ubachs et al., 2004; 2006; Abdul-Baqi et al., 2005). However, fundamental interfacial properties such as the cohesive/adhesive strength for different physical interfaces are lacking and these parameters in such models are mostly assumed values and their identification depends entirely on adequate and full-field experimental analysis of the deformation of solder materials. Therefore, the development of non-classical (but continuum) microstructural damage models that are amenable to engineering-design of solder materials is desirable. Tang and Basaran (2003) have proposed a damage mechanics based constitutive model

for solder alloys where damage evolution is described in terms of entropy production. Gomez and Basaran (2005) modified Tang and Basaran (2003) damage model to incorporate a material length scale through the use of Fleck and Hutchinson (1993) strain gradient plasticity theory. Ubachs et al. (2006) formulated a finite deformation viscoplastic-damage model for solder alloys where a strong nonlocal damage parameter is presented based on the implicit gradient-dependent damage model of Peerlings et al. (1996).

It is well established in the literature that the final failure of a solder joint is preceded by inhomogeneities in the deformation of the joint at relatively early stages, which cause an initial evolution of defects in the virgin material state in the form of localised zones, such as the nucleation of certain amount of cracks, voids, dislocations, and shear bands. Those localised defects of plasticity and damage induced in the material structure along with the subsequent defects that occur during deformation process leads to a heterogeneous (non-uniform) material behaviour. Further loading of materials of this type will cause failure mechanisms to occur at localised zones of plasticity and damage. In those localised zones, many defects may undergo irreversible growth; coalescence of pre-existing cracks and voids may occur; propagation of dislocations may proceed; and new defects may nucleate and their ultimate coalescence results in failure. Moreover, intensive interaction mechanisms of the evolved defects may take place at those localised zones; such as dislocation-dislocation interaction, microdamage-microdamage interaction, crack dominated-dislocation interaction, dislocation dominated-crack interaction, dislocation/crack-grain boundary interaction, etc. Therefore, as the plasticity and damage defects localise over narrow regions of the continuum, the characteristic length scale governing the variations of those defects and their average interactions over multiple length scales falls far below the scale of the local state variables of classical plasticity and damage theories used to describe the response of the continuum. This leads to the loss of the statistical homogeneity in the representative volume element (RVE) and causes strong scale effects; in such a way that all the macroscopic response functions of interest (e.g. the Helmholtz free energy, Ψ ; the dissipation potential, Π ; the Cauchy stress tensor, σ ; the small strain tensor, ϵ ; the stiffness tensor; E ; etc.) are sensitive to the distribution, size, and orientation of the micro-, meso- and macro-structural defects within the RVE. The plasticity and damage evolution processes are, therefore, statistically inhomogeneous at the macroscale level (at the RVE scale). This suggests that the macroscopic inelastic deformations and failure are governed by mechanisms at different scale levels (nonlocality) which gives rise to the gradient effects. Thus, the gradient effect is important when the characteristic dimension of the plastic and/or damage deformation zone is of the same order as the material intrinsic length scale, which is in the order of microns for commonly used materials (Gao et al., 1999a; 1999b; Abu Al-Rub and Voyiadjis, 2004a; 2004b, Abu Al-Rub, 2007). For example, micro-cracks and micro-voids are observed on a mesolevel with length scale $0.1-10\mu m$ affecting strongly the material behaviour on the macrolevel with length scale $\geq 100\mu m$.

The use of the nonlocal (gradient type or integral type) theories is increasing now days in order to appropriately overcome this problem, which takes into account the influence of the n^{th} nearest neighbour of the material points or the long-range microstructural interactions. The enhanced nonlocal theories formulate a constitutive framework on the continuum level that is used to bridge the gap between the micromechanical theories and the classical (local) continuum. They are successful in

explaining the size effects encountered at the micron and submicron scales (for example see Fleck et al., 1994; Aifantis, 1995; Fleck and Hutchinson, 1997; 2001; Gao et al., 1999a; 1999b; Acharya and Bassani, 2000; Gurtin, 2002; 2003; Taylor et al., 2002; Abu Al-Rub and Voyiadjis, 2004a; 2004b; Abu Al-Rub, 2007; Abu Al-Rub et al., 2007; and the references cited therein) and in preserving the well-posedness of the (initial) boundary value problems governing the solution of material instability triggering strain localisation (see for example Aifantis, 1984; Bazant et al., 1984; Pijaudier-Cabot and Bazant, 1987; Bazant and Pijaudier-Cabot, 1988; Lasry and Belytschko, 1988; Bazant and Ozbolt, 1990; de Borst and Mühlhaus, 1992; Zbib and Aifantis, 1992; de Borst et al., 1993; de Borst and Pamin, 1996; Peerlings et al., 1996; Fremond and Nedjar, 1996; de Borst et al., 1999; Bammann et al., 1999; Ganghoffer et al., 1999; Askes et al., 2000; Kuhl et al., 2000; Lacey et al., 2000; Abu Al-Rub and Voyiadjis, 2005; Voyiadjis and Abu Al-Rub; 2006; and many other references cited therein). This success is due to the incorporation of an intrinsic material length scale parameter in the constitutive relations of classical (local) theories of damage and plasticity such that this length scale parameter adequately captures the size of localised zones in heterogeneous media.

However, it should be emphasised that although existing rate-dependent (viscoplasticity and/or viscodamage) theories regularise the solution by introducing implicitly a length scale through the viscosity parameter, (e.g. Perzyna, 1988; Needleman, 1988; Wang et al., 1996; 1998; Dornowski and Perzyna, 2000; Glema et al., 2000; Voyiadjis and Abed, 2005), the numerical results still show a mesh dependency (Voyiadjis et al., 2006) especially at low strain rates. Moreover, rate-dependent theories do not adequately capture scale effects at the micron and submicron length scales (Voyiadjis et al., 2004; Voyiadjis and Abu Al-Rub, 2006). Therefore, both the nonlocal gradient-dependent approach and rate-dependency will be incorporated in the current model.

The objective of the present paper is to develop a consistent and systematic gradient-enhanced coupled thermo-viscoplastic-viscodamage model for microelectronic solder alloys. Furthermore, a general thermodynamic framework for the modelling of heterogeneous media that assesses a strong coupling between thermo-viscoplasticity and thermo-viscodamage evolution in solder materials is developed with consideration of the anisotropic distribution of discontinuities at the macro and meso scales. The constitutive equations are derived from the first and second laws of thermodynamics, the expression of Helmholtz free energy, the Clausius-Duhem inequality, the maximum dissipation principle, generalised normality, and the thermo-mechanical heat balance equation. The nonlocality is introduced here through the tensorial damage variable. The local damage variable and its second-order gradient (Laplacian) are considered independent internal state variables such that each gives different physical interpretations that guide one to different evolution equations and allowing one to computationally introduce independently the influence of the macro and meso scales.

The outline of this paper is as follows: in Section 2, the local effective (undamaged) configuration concept in continuum damage mechanics is extended to a nonlocal one. In Section 3, we outline a general thermodynamic framework for the nonlocal elasto-viscoplastic and viscodamage material behaviour with thermal effects using the gradient-dependent theory. In Sections 4 and 5 the rate-type constitutive stress-strain relation and the thermo-mechanical heat balance equation are derived, respectively. Finally, in Section 6 the physical interpretation and identification of the incorporated intrinsic material length scale are discussed.

2 Nonlocal fictitious undamaged configuration

Continuum damage mechanics based on the effective stress space was introduced by Kachanov (1958) and later by Rabotnov (1968) who were the first to introduce for the isotropic damage case a one-dimensional variable, ϕ , which may be interpreted as the effective surface density of microdamage per unit volume (Abu Al-Rub and Voyiadjis, 2003). The effective stress tensor, $\bar{\sigma}$, can be expressed in terms of the nominal stress, σ , as follows:

$$\bar{\sigma}_{ij} = \frac{\sigma_{ij}}{1 - \phi} \quad (1)$$

However, the microstructure of solder materials is heterogeneous due to the interplay of characteristic lengths including sizes or spacing of static microstructural features (e.g. solder constituents) and evolving defect clusters. As traditional continuum mechanics does not contain characteristic lengths, the use of the nonlocal concept is required in order to introduce a microstructural characteristic length scale and in order to introduce long-range (nonlocal) microstructural interactions where the stress response at a material point is assumed to depend on the state of its neighbourhood in addition to the state of the material point itself. Moreover, equation (1) is a mathematical definition of the effective stress although it may be interpreted as the average stress acting on an effective area of the material (e.g. Kachanov, 1986; Lemaitre and Chaboche, 1990; Lemaitre, 1992; Krajcinovic, 1996; Voyiadjis and Kattan, 1999). In order to give it a general physical meaning, it is necessary to use the corresponding damage-free material (i.e. virgin material) in the mesoscale to represent the ‘effective’ concept of equation (1) for a macroscopically damaged material. Thus, a proper correlating hypothesis between two material scale levels can be obtained through using the nonlocal damage variable $\hat{\phi}$ (e.g. Pijaudier-Cabot and Bazant, 1987; Bazant and Pijaudier-Cabot, 1988; Voyiadjis et al., 2003; 2004; Abu Al-Rub and Voyiadjis, 2006). It is then important to emphasise that Kachanov’s definition given by equation (1) can be generalised to a nonlocal one, for the case of isotropic damage and under a general state of stress, as follows:

$$\bar{\sigma}_{ij} = \frac{\sigma_{ij}}{1 - \hat{\phi}} \quad (2)$$

The nonlocal damage variable $\hat{\phi}$ can be defined at position \mathbf{x} as the weighted average of its local counterpart ϕ over a surrounding volume V at a small distance $\|\boldsymbol{\zeta}\| \leq \ell$ from the considered point (Kroner, 1967; Pijaudier-Cabot and Bazant, 1987), such that:

$$\hat{\phi}(\mathbf{x}) = \frac{1}{V_r(\mathbf{x})} \int_V \mathbf{h}(\boldsymbol{\zeta}) \phi(\mathbf{x} + \boldsymbol{\zeta}) dV \quad (3)$$

in which $V_r(\mathbf{x}) = \int_V \mathbf{h}(\boldsymbol{\zeta}) dV$. In the above and subsequent equations, the superimposed hat designates a spatial nonlocal operator, ℓ is an intrinsic characteristic length scale parameter, and $\mathbf{h}(\boldsymbol{\zeta})$ is a weight function that decays smoothly with distance. The

evolution of the local variable ϕ in equation (3) can be approximated by a Taylor expansion around $\boldsymbol{\zeta} = 0$, such that:

$$\phi(\mathbf{x} + \boldsymbol{\zeta})\big|_{\boldsymbol{\zeta}=0} = \phi(\mathbf{x}) + \nabla\phi(\mathbf{x})\boldsymbol{\zeta} + \frac{1}{2}\nabla^2\phi(\mathbf{x})\boldsymbol{\zeta}\boldsymbol{\zeta} + \frac{1}{3!}\nabla^3\phi(\mathbf{x})\boldsymbol{\zeta}\boldsymbol{\zeta}\boldsymbol{\zeta} + \dots \quad (4)$$

where ∇^i denotes the i -th order gradient operator. Substituting equation (4) into equation (3) and assuming only an isotropic influence of the averaging equation (i.e. the nonlocal weighting function \mathbf{h} is isotropic such that $\mathbf{h}(\boldsymbol{\zeta}) = \mathbf{I} h(\boldsymbol{\zeta})$, where \mathbf{I} is an identity tensor), the integrals of the odd terms vanish. However, the weighting function $\mathbf{h}(\boldsymbol{\zeta})$ may be suitably substituted by another tensor in order to induce nonlocal anisotropic behaviour (Voyiadjis et al., 2004), which is not considered in the current study. Furthermore, neglecting terms after the quadratic term leads to the following expression for the nonlocal damage variable $\widehat{\phi}$:

$$\widehat{\phi} = \frac{1}{V_r} \int_V h(\boldsymbol{\zeta})\phi(\mathbf{x})dV + \frac{1}{2V_r} \int_V h(\boldsymbol{\zeta})\nabla^2\phi(\mathbf{x})\boldsymbol{\zeta}\boldsymbol{\zeta}dV \quad (5)$$

By setting $\frac{1}{V_r} \int_V h(\boldsymbol{\zeta})dV = 1$ (Pijaudier-Cabot and Bazant, 1987) and substituting the following Gaussian error distribution function for $h(\boldsymbol{\zeta})$

$$h(\boldsymbol{\zeta}) = \exp\left(-\|\boldsymbol{\zeta}\|^2/2\ell^2\right) \quad (6)$$

in equation (5) and integrating the result from $-\infty$ to ∞ leads to

$$\widehat{\phi} = \phi + \frac{1}{2}\ell^2\nabla^2\phi \quad (7)$$

Therefore, an attempt is made here to account for the effect of non-uniform distribution of microdefects on the overall macroscale response by assuming the thermo-elastic Helmholtz free energy density, Ψ , to depend not only on the macroscopic response associated with the internal state variable ϕ , but also on its macroscopic spatial second-order gradient (or Laplacian), $\nabla^2\phi$. This will be detailed in the subsequent sections of this paper.

Many researchers tend to adopt the traditional simple isotropic scalar damage variable, $(1-\phi)$, which is generalised here as $(1-\widehat{\phi})$ where $\widehat{\phi}$ is a nonlocal quantity, to model the material microdamage mechanisms, in which all components of the material stiffness are degraded by the same scalar damage parameter, ϕ (or $\widehat{\phi}$ in this work). However, to ensure a more general formulation of the principles of continuum damage mechanics, the case of anisotropic damage will be considered in this work. In this case, different levels of damage are related to the principal directions, and thus a simple scalar damage parameter is no longer sufficient to quantify damage in all directions. Instead, the anisotropic phenomenon of the microdamage (i.e. microcracks and microvoids) distribution in the solder material is interpreted using a nonlocal symmetric second-order damage tensor, $\widehat{\boldsymbol{\phi}}$, as a generalisation to equation (7), such that

$$\widehat{\phi}_{ij} = \phi_{ij} + \frac{1}{2} \ell^2 \nabla^2 \phi_{ij} \quad (8)$$

whose evolution will be defined later.

The linear elastic constitutive equations for the damaged material are expressed by equation (1). That is, the damaged material is modelled using the constitutive laws of the effective undamaged material in which the Cauchy stress tensor, $\boldsymbol{\sigma}$, can then be replaced by the effective stress tensor, $\bar{\boldsymbol{\sigma}}$ (Murakami and Ohno, 1981) as follows:

$$\bar{\sigma}_{ij} = \widehat{M}_{ijkl} \sigma_{kl} \quad (9)$$

where \widehat{M} is the nonlocal fourth-order damage-effect tensor. The tensor \widehat{M} , which is then expressed in terms of $\widehat{\phi}$, characterises the notion of non-uniform distribution and interaction of microdamage (microcracks and microvoids) over multiple length scales at which first and second nearest neighbour effects of nonlocal character are significant, similar to the homogenisation theory. Many different expressions for \widehat{M} have been proposed in the literature in order to symmetrise the effective stress tensor, $\bar{\boldsymbol{\sigma}}$. A comprehensive review of the most widely used expressions are presented by Voyiadjis and Kattan (1999). The following expression for \widehat{M} , which is proposed by Cordebois and Sidoroff (1979) for the local concept of damage, is used here due to its attractiveness in the mathematical formulation, such that:

$$\widehat{M}_{ijkl} = 2 \left[\left(\delta_{ik} - \widehat{\phi}_{ik} \right) \delta_{jl} + \delta_{ik} \left(\delta_{jl} - \widehat{\phi}_{jl} \right) \right]^{-1} \quad (10)$$

where δ_{ij} is the Kronecker delta. Note that the fourth-order tensor \widehat{M} exhibits the major symmetries only (i.e. $\widehat{M}_{ijkl} = \widehat{M}_{jilk}$).

In order to derive the transformation relations between the damaged and the hypothetical undamaged (effective configuration) states of the material, the elastic energy equivalence hypothesis (Sidoroff, 1981) is utilised here. This hypothesis assumes that the elastic strain energy density in the effective configuration is equal to the corresponding one in the damaged configuration such that the effective elastic strain tensor, $\bar{\boldsymbol{\epsilon}}^e$, can be related to the nominal elastic strain tensor, $\boldsymbol{\epsilon}^e$, by:

$$\bar{\epsilon}_{ij}^e = \widehat{M}_{ijkl}^{-1} \epsilon_{kl}^e \quad (11)$$

Furthermore, the elastic-damage stiffness, \boldsymbol{E} , can be expressed as follows (e.g. Voyiadjis et al., 2004; Abu Al-Rub and Voyiadjis, 2003; 2006):

$$E_{ijkl} = \widehat{M}_{imjn}^{-1} \bar{E}_{mnpq} \widehat{M}_{pkql}^{-1} \quad (12)$$

where

$$\widehat{M}_{ijkl}^{-1} = \frac{1}{2} \left[\left(\delta_{ik} - \widehat{\phi}_{ik} \right) \delta_{jl} + \delta_{ik} \left(\delta_{jl} - \widehat{\phi}_{jl} \right) \right] \quad (13)$$

and $\bar{\mathbf{E}}$ is the fourth-order elastic moduli tensor, which is anisotropic for solder materials. However, the elastic-damage stiffness, \mathbf{E} , exhibits the major and minor symmetries similar to the elastic stiffness, $\bar{\mathbf{E}}$.

Using equation (8) into equation (13) one can rewrite $\widehat{\mathbf{M}}^{-1}$ as:

$$\widehat{\mathbf{M}}_{ijkl}^{-1} = \mathbf{M}_{ijkl}^{-1} + \frac{1}{2} \ell^2 \nabla^2 \mathbf{M}_{ijkl}^{-1} \quad (14)$$

where

$$\mathbf{M}_{ijkl}^{-1} = \frac{1}{2} \left[(\delta_{ik} - \phi_{ik}) \delta_{jl} + \delta_{ik} (\delta_{jl} - \phi_{jl}) \right] \quad (15)$$

and

$$\nabla^2 \mathbf{M}_{ijkl}^{-1} = -\frac{1}{2} \left(\nabla^2 \phi_{ik} \delta_{jl} + \delta_{ik} \nabla^2 \phi_{jl} \right) \quad (16)$$

For small strain problems an additive decomposition of the rate of the total strain tensor, $\dot{\boldsymbol{\varepsilon}}$, can be assumed with $\dot{\boldsymbol{\varepsilon}}^e$ being the elastic component and $\dot{\boldsymbol{\varepsilon}}^{vi}$ being the corresponding viscoinelastic component such that:

$$\dot{\boldsymbol{\varepsilon}}_{ij} = \dot{\boldsymbol{\varepsilon}}_{ij}^e + \dot{\boldsymbol{\varepsilon}}_{ij}^{vi} \quad (17)$$

The elastic strain, $\boldsymbol{\varepsilon}^e$, is the reversible part of the total strain which is attributed to the elastic distortions, cracks closure, and voids contraction upon unloading (but not healing), while the viscoinelastic strain, $\boldsymbol{\varepsilon}^{vi}$, designates the irreversible part of the total strain which is attributed to viscoplastic distortions and viscodamage strains characterised by the lack of cracks closure and voids contraction which cause permanent deformations. The lack of cracks closure and voids contraction can be due to the constraints set up by the interacting (micro)-cracks, (micro)-voids, dislocation movements, and external/internal interfaces.

3 Thermodynamic framework

3.1 Internal state variables

In this work, thermal, elastic, viscoplastic (rate-dependent plasticity), and viscodamage (rate-dependent damage or creep damage) material behaviour is considered. This means that the stress path, strain rate, temperature material dependence, and the nonlinear material response are all considered in this work. Thus, the dependent constitutive variables are functions of the elastic strain tensor, $\boldsymbol{\varepsilon}^e$, the absolute temperature, T , the temperature gradient vector, $\nabla_i T$, and n_{int} - of phenomenological internal state variables, \mathcal{N}_k ($k = 1, \dots, n_{\text{int}}; n_{\text{int}} \geq 1$). Hence, within the thermodynamic framework and considering the assumption of infinitesimal displacements/strain relationships, the Helmholtz free energy density function can be written as (Coleman and Gurtin, 1967; Lubliner, 1990; Lemaitre and Chaboche, 1990; Doghri, 2000):

$$\Psi = \tilde{\Psi} \left(\boldsymbol{\varepsilon}_{ij}^e, T, \nabla_i T; \mathcal{N}_k \right) \quad (18)$$

Since the main objective is to develop the rate type constitutive equations for a thermo-viscoplastic and thermo-viscodamage material, the effects of strain rate, viscoplastic strain hardening/softening, viscodamage strain hardening/softening, micro-damage mechanisms, and thermo-mechanical coupling have to be considered. In order to describe such mechanisms, a finite set of internal state variables, \mathfrak{N}_k , representing either a scalar or a tensorial variable are assumed, such that:

$$\mathfrak{N}_k = \tilde{\mathfrak{N}}_k \left(p, \alpha_{ij}, r, \phi_{ij}, \nabla^2 \phi_{ij} \right) \quad (19)$$

where p denotes the accumulative equivalent viscoplastic strain, α denotes the flux of the residual stress (backstress); and r denotes the accumulative damage. The variables p and α are associated with isotropic hardening and kinematic hardening in the viscoplastic flow process, respectively. Note that ϕ and $\nabla^2 \phi$, which appear in the nonlocal expression in equation (7), are assumed independent internal state variables where each represent a distinct length scale effect. Both $\nabla^2 \phi$ and ϕ play as an explicit link between the damage evolution at the mesoscale and the behaviour of the homogenous equivalent material at the macroscale. Furthermore, the assumed dependence of the Helmholtz free energy on the distinct variables $\nabla^2 \phi$ is also motivated by the necessity to include an intrinsic material length scale measure into the constitutive equations that can regularise the numerical computations and make it insensitive to discretisation (e.g. mesh-independent results in the finite element analysis) [see Abu Al-Rub and Voyiadjis (2005) and the many references cited therein about this topic]. The damage variable $\nabla^2 \phi$ reflects the long-range microstructural deterioration due to nucleation, growth, and coalescence of voids, cavities, and microcracks. It may also account for internal embedded crack-tip stress variations introduced by crack pile-ups and, moreover, for the lack of a proper statistical distribution of microcracks and microvoids due to viscodamage localisation.

It is noteworthy that the state variables in equation (19) are introduced in the Helmholtz free energy density in order to provide sufficient details of the deformation defects (cracks, voids, mobile and immobile dislocation densities) and their interactions, and to properly (i.e. physically) characterise the material microstructural behaviour. Those variables will provide an adequate characterisation of these defects in terms of size, orientation, distribution, spacing, interaction among defects, and so forth. Moreover, introducing the higher-order damage variable $\nabla^2 \phi$ in the Helmholtz free energy allows the two different physical phenomena in the meso- and macro-scales to be identified separately with different evolution equations. This can be simply argued based on the concepts of statistically stored dislocations (SSDs) and geometrically necessary dislocations (GNDs) in polycrystalline metals. Standard micromechanical modeling of the inelastic material behaviour of metallic single crystals and polycrystals is commonly based on the premise that resistance to glide is due mainly to the random trapping of mobile dislocations during locally homogeneous deformation. Such trapped dislocations are commonly referred to as SSDs, and act as obstacles to further dislocation motion, resulting in hardening. As anticipated in the work of Ashby (1970), an additional contribution to the density of immobile dislocations and so to hardening can arise when the continuum length scale approaches that of the dominant microstructural features (e.g. mean spacing between inclusions relative to the inclusion size when considering a

microstructure with dispersed inclusions, size of the plastic process zone at the front of the crack tip, the mean spacing between dislocations, the grain size, etc.). Indeed, in this case, the resulting deformation incompatibility between, e.g., ‘hard’ substrate and a ‘soft’ solder material, is accommodated by the development of GNDs which maintain the plastic deformation compatibilities (continuity) within the polycrystal (or various components of the material) caused by non-uniform dislocation slip. SSDs are related to the effective plastic strain which drives strain hardening whereas the gradient of the effective plastic strain is related to GNDs. The evolution of GND density cannot be simply obtained by taking the gradient of SSDs density. Each represents a different physical mechanism that guides one to different evolution equations. Therefore, one can assume that the gradient of the state variable is independent of the state variable itself.

The determination of the evolution of the assumed internal state variables is the main challenge of the modern constitutive modeling. This can be effectively achieved, so far, through the thermodynamic principles for the development of a continuum thermo-elasto-viscoplastic and thermo-viscodamage-based model. That is, use is made of the balancing laws, the conservation of mass, linear and angular momenta, and the first and second laws of thermodynamics. Those fundamental laws of continuum mechanics can be written as follows (Coleman and Gurtin, 1967; Lubliner, 1990; Lemaitre and Chaboche, 1990; Doghri, 2000):

- 1 conservation of mass

$$\dot{\rho} + \rho v_{i,i} = 0 \quad (20)$$

- 2 balance of linear momentum

$$\sigma_{ij,j} + \rho b_i = \rho \dot{v}_i \quad (21)$$

- 3 balance of moment of momentum

$$\sigma_{ij} = \sigma_{ji} \quad (22)$$

- 4 conservation of energy (first law of the thermodynamics)

$$\rho \dot{e} = \sigma_{ij} \dot{\epsilon}_{ij} + \rho r_{ext} - q_{i,i} \quad (23)$$

- 5 Clausius-Duhem inequality

$$\sigma_{ij} \dot{\epsilon}_{ij} - \rho \left(\dot{\Psi} + \eta \dot{T} \right) - \frac{1}{T} q_i \nabla_i T \geq 0 \quad (24)$$

where ρ , v , b , e , r_{ext} , η , and q are the mass density, the velocity vector, the body force vector, the internal energy density, the density of external heat, the specific entropy, and the heat flux vector, respectively. Meanwhile, e , Ψ , T , and η are related by:

$$\Psi = e - T\eta \quad (25)$$

For the purpose of describing the viscoinelastic material behaviour, an additive decomposition of the specific free energy function, Ψ , into thermo-elastic, thermo-viscoplastic, and thermo-viscodamage parts is assumed here, such that:

$$\begin{aligned} \tilde{\Psi}(\varepsilon_{ij}^e, T, \nabla_i T; \mathfrak{N}_k) &= \tilde{\Psi}^{te}(\varepsilon^e, T, \nabla_i T, \phi_{ij}, \nabla^2 \phi_{ij}) + \tilde{\Psi}^{vp}(T, \nabla_i T, p, \alpha_{ij}) \\ &+ \tilde{\Psi}^{vd}(T, \nabla_i T, r, \phi_{ij}, \nabla^2 \phi_{ij}) \end{aligned} \quad (26)$$

where Ψ^{te} is the thermo-elastic stored energy, while Ψ^{vp} and Ψ^{vd} are the energies stored due to material hardening in viscoplasticity and viscodamage deformation mechanisms, respectively. It should be noted that equation (26) is a partially decoupled form of the specific free energy Ψ . There is no state coupling between viscoplasticity and elasticity, but the state coupling of viscodamage with the thermo-elastic and thermo-viscoplastic stored energies strongly appears in the above decomposition. In particular, the damage variables ϕ and $\nabla^2 \phi$ appear in all portions of Ψ and the other internal state variables are expressed in the current, deformed, and damaged configurations. ϕ and $\nabla^2 \phi$ appear implicitly in Ψ^{vp} through the evolution of its set of internal state variables as will be shown in the subsequent developments.

According to the definition given above for Ψ , the time derivative of equation (18) with respect to its internal state variables is given by:

$$\dot{\Psi} = \frac{\partial \Psi}{\partial \varepsilon_{ij}^e} \dot{\varepsilon}_{ij}^e + \frac{\partial \Psi}{\partial T} \dot{T} + \frac{\partial \Psi}{\partial \nabla_i T} \nabla_i \dot{T} + \frac{\partial \Psi}{\partial \mathfrak{N}_k} \dot{\mathfrak{N}}_k \quad (27)$$

where from equation (19), one can also write

$$\frac{\partial \Psi}{\partial \mathfrak{N}_k} \dot{\mathfrak{N}}_k = \frac{\partial \Psi}{\partial p} \dot{p} + \frac{\partial \Psi}{\partial \alpha_{ij}} \dot{\alpha}_{ij} + \frac{\partial \Psi}{\partial r} \dot{r} + \frac{\partial \Psi}{\partial \phi_{ij}} \dot{\phi}_{ij} + \frac{\partial \Psi}{\partial \nabla^2 \phi_{ij}} \nabla^2 \dot{\phi}_{ij} \quad (28)$$

Substituting equations (27) and (28) into the Clausius-Duhem inequality, equation (24), along with equation (17), one obtains the following thermodynamic constraint:

$$\left(\sigma_{ij} - \rho \frac{\partial \Psi}{\partial \varepsilon_{ij}^e} \right) \dot{\varepsilon}_{ij}^e + \sigma_{ij} \dot{\varepsilon}_{ij}^{vi} - \rho \left(\frac{\partial \Psi}{\partial T} + \eta \right) \dot{T} - \rho \frac{\partial \Psi}{\partial \nabla_i T} \nabla_i \dot{T} - \rho \frac{\partial \Psi}{\partial \mathfrak{N}_k} \dot{\mathfrak{N}}_k - \frac{q_i}{T} \nabla T \geq 0 \quad (29)$$

Assuming that the axiom of entropy production holds, then the above inequality equation results in the following thermodynamic state laws:

$$\sigma_{ij} = \rho \frac{\partial \Psi}{\partial \varepsilon_{ij}^e}; \quad \eta = -\frac{\partial \Psi}{\partial T}; \quad \frac{q_i}{T} = \rho \frac{\partial \Psi}{\partial \nabla_i T}; \quad \Sigma_k = \rho \frac{\partial \Psi}{\partial \mathfrak{N}_k} \quad (k=1, \dots, 5) \quad (30)$$

The above equations describe the relations between the state variables (observable and internal) and their associated thermodynamic conjugate forces. These thermodynamic forces conjugate to their state variables are $\Sigma_k = \{ R, \mathbf{X}, K, \mathbf{Y}, \mathbf{Y}^g \}$ ($k=1, \dots, 5$) are the conjugate forces corresponding to the viscoplastic and viscodamage internal state variables $\mathfrak{N}_k = \{ p, \boldsymbol{\alpha}, r, \boldsymbol{\phi}, \nabla^2 \boldsymbol{\phi}, \}$ ($k=1, \dots, 5$), respectively. The stress σ is the Cauchy stress tensor, R is the plasticity isotropic hardening stress, \mathbf{X} is the plasticity backstress, and K is the damage hardening function. The damage driving forces \mathbf{Y} and \mathbf{Y}^g are measures of the elastic-damage changes in the internal structure resulting from crack closure and voids contraction during the unloading process, which can be

interpreted as the energy release rates (Abu Al-Rub and Voyiadjis, 2003). A summary of the internal state variables and their conjugate forces is presented in Table 1.

Table 1 Thermodynamic state variables and their corresponding conjugate forces

<i>State variables</i>		<i>Associated conjugates</i>
<i>Observable</i>	<i>Internal</i>	
ε		σ
T		η
	ε^e	σ
	ε^{vi}	$-\sigma$
	∇T	\mathbf{q}
	$\aleph_1 = p, \aleph_2 = \boldsymbol{\alpha}$	$\Sigma_1 = R, \Sigma_2 = \mathbf{X}$
	$\aleph_3 = r, \aleph_4 = \boldsymbol{\phi}, \aleph_5 = \nabla^2 \boldsymbol{\phi}$	$\Sigma_3 = K, \Sigma_4 = -Y, \Sigma_5 = -Y^g$

The state laws of the assumed internal state variables, equation (30)₄, are explicitly expressed for thermo-viscoplasticity and thermo-viscodamage, respectively, as follows:

$$R = \rho \frac{\partial \Psi^{vp}}{\partial p}; \quad X_{ij} = \rho \frac{\partial \Psi^{vp}}{\partial \alpha_{ij}} \quad (31)$$

$$K = \rho \frac{\partial \Psi^{vd}}{\partial r}; \quad -Y_{ij} = \rho \frac{\partial \Psi^{te}}{\partial \phi_{ij}}; \quad -Y_{ij}^g = \rho \frac{\partial \Psi^{te}}{\partial \nabla^2 \phi_{ij}} \quad (32)$$

Substituting equations (30)–(32) into equation (29), one reduces the Clausius-Duhem inequality in order to express the fact that the dissipation energy, Π , is necessarily positive as follows:

$$\Pi = \sigma_{ij} \dot{\varepsilon}_{ij}^{vi} - \Pi_{\text{int}} - q_i \left(\frac{\nabla_i T}{T} + \frac{\nabla_i \dot{T}}{\dot{T}} \right) \geq 0 \quad (33)$$

where the internal viscoinelastic dissipation energy, Π_{int} , can be written as:

$$\Pi_{\text{int}} = \sum_{k=1}^5 \sum_k \dot{\aleph}_k = R\dot{p} + X_{ij} \dot{\alpha}_{ij} + K\dot{r} - Y_{ij} \dot{\phi}_{ij} - Y_{ij}^g \nabla^2 \dot{\phi}_{ij} \geq 0 \quad (34)$$

To this end, the following split is proposed: $n_{\text{int}} = n_{\text{int}}^{vp} + n_{\text{int}}^{vd}$, where n_{int}^{vp} and n_{int}^{vd} refer to the number of internal state variables related to thermo-viscoplastic and thermo-viscodamage (irreversible that may occur in every material) effects, respectively. In this context, this assumption leads to rewriting the dissipation energy, Π , as the summation of dissipations due to mechanical (viscoplasticity and viscodamage) and thermal (heat conduction) effects as:

$$\Pi = \Pi^{vp} + \Pi^{vd} + \Pi^{th} \geq 0 \quad (35)$$

where

$$\Pi^{vp} = \sigma_{ij} \dot{\epsilon}_{ij}^{vi} - R\dot{p} - X_{ij} \dot{\alpha}_{ij} \geq 0 \quad (36)$$

$$\Pi^{vd} = -K\dot{r} + Y_{ij} \dot{\phi}_{ij} + Y_{ij}^g \nabla^2 \dot{\phi}_{ij} \geq 0 \quad (37)$$

$$\Pi^{th} = -q_i \left(\frac{\nabla_i T}{T} + \frac{\nabla_i \dot{T}}{\dot{T}} \right) \geq 0 \quad (38)$$

This result requires all viscoelastic work to dissipate away as heat, except for that energy which is stored because of the rearrangement of the material internal structure. Although, we write the thermo-viscoelastic dissipation function Π in the decoupled form as shown by equation (35); however, this does not imply that the corresponding physical mechanisms are decoupled. Strong coupling does occur in the viscoplastic potential given by equation (36) between viscoplasticity and viscodamage since the conjugate forces and their associated fluxes are expressed in the current, deformed, and damaged configuration of the material. Hence, two additive damage mechanisms are introduced in the dissipation function, equation (35); one mechanism is coupled with viscoplasticity and the other occurs independent of viscoplastic deformation.

It is clearly seen that the definition of Ψ and consequently of $\dot{\Sigma}_k$ ($k=1, \dots, 5$) are essential features of the formulation in order to describe the thermo-mechanical/microstructural behaviour of the material involved in the deformation process. The associative evolution laws of ϵ^{vi} and ϕ can be obtained by utilising the calculus of several variables with Lagrange multipliers $\dot{\lambda}^{vp}$ and $\dot{\lambda}^{vd}$. The inelastic dissipation function $\Pi^I = \Pi^{vp} + \Pi^{vd}$, equation (35), is subjected to the two constraints, namely the viscoplastic potential F and the viscodamage potential G (Voyiadjis and Kattan, 1999), such that:

$$\Omega = \Pi^{vp} + \Pi^{vd} - \dot{\lambda}^{vp} F - \dot{\lambda}^{vd} G \quad (39)$$

One now makes use of the maximum viscoelastic dissipation principle (Simo and Honein, 1990; Simo and Hughes, 1998), which states that the actual state of the thermodynamic forces is that which maximises the inelastic dissipation function over all other possible admissible states. Therefore, one can maximise the objective function Ω by using the necessary conditions as follows:

$$\frac{\partial \Omega}{\partial \sigma_{ij}} = 0 \quad \text{and} \quad \frac{\partial \Omega}{\partial \Sigma_k} = 0 \quad (k=1, \dots, 5) \quad (40)$$

Substitution of equation (39) into equation (40)₁ along with equations (36) and (37) yields the thermodynamic law corresponding to the evolution of the viscoelastic strain rate, $\dot{\epsilon}^{vi}$, as follows:

$$\dot{\epsilon}_{ij}^{vi} = \dot{\epsilon}_{ij}^{vp} + \dot{\epsilon}_{ij}^{vd} \quad (41)$$

where

$$\dot{\boldsymbol{\varepsilon}}_{ij}^{vp} = \dot{\lambda}^{vp} \frac{\partial F}{\partial \sigma_{ij}}; \quad \dot{\boldsymbol{\varepsilon}}_{ij}^{vd} = \dot{\lambda}^{vd} \frac{\partial G}{\partial \sigma_{ij}} \quad (42)$$

such that $\dot{\boldsymbol{\varepsilon}}^{vp}$ designates the viscoplastic strain rate and $\dot{\boldsymbol{\varepsilon}}^{vd}$ is the viscodamage strain rate. Therefore, equation (42)₁ is the conventional viscoplasticity flow rule, while equation (42)₂ describes the irreversible strain rate due to microdamage growth characterised by the lack of micro-cracks closure and micro-voids contraction which causes permanent deformations. The lack of micro-cracks closure and micro-voids contraction can be due to the constraints set up by the interacting micro-cracks, micro-voids, dislocation movements, and external/internal interfaces (Abu Al-Rub and Voyiadjis, 2003). Thus, if the material is damage-free, only $\dot{\boldsymbol{\varepsilon}}^{vp}$ is used to describe the irreversible thermodynamic process evolving in time. Moreover, equation (42) enhances strong coupling between plasticity and damage mechanisms.

On the other hand, equation (40)₂ yields the following evolution laws for the internal state variables in equation (19):

$$\dot{p} = -\dot{\lambda}^{vp} \frac{\partial F}{\partial R}; \quad \dot{\alpha}_{ij} = -\dot{\lambda}^{vp} \frac{\partial F}{\partial X_{ij}} \quad (43)$$

$$\dot{r} = -\dot{\lambda}^{vd} \frac{\partial G}{\partial K}; \quad \dot{\phi}_{ij} = \dot{\lambda}^{vd} \frac{\partial G}{\partial Y_{ij}}; \quad \nabla^2 \dot{\phi}_{ij} = \nabla^2 \dot{\lambda}^{vd} \frac{\partial G}{\partial Y_{ij}^s} + \dot{\lambda}^{vd} \nabla^2 \frac{\partial G}{\partial Y_{ij}^s} \quad (44)$$

where the last term in equation (44), $\nabla^2 \partial_{Y^s} G \approx 0$ can be neglected (Voyiadjis et al., 2004).

In order to obtain explicit evolution laws for equations (41)–(44), one needs to define the potentials F and G , which will be presented in Subsection 3.3.

3.2 Specific free energy function

As it is clearly seen in the previous section, the complexity of a model is directly determined by the form of the Helmholtz free energy Ψ and by the number of conjugate pairs of variables. Therefore the definition of Ψ constitutes a crucial point of the formulation since it is the basis for the derivation of all the constitutive equations to be described in what follows. It is possible to decouple the Helmholtz free energy into a potential function for each of the internal state variable in such a way that an analytical expression for the thermodynamic potential is given as a quadratic form of its internal state variables. However, coupling is possible in the viscoplastic potential or the viscodamage potential if they depend on more than one variable, which makes the evolution equations more complex. Therefore, the thermo-elastic energy, Ψ^{te} , can be postulated as follows:

$$\rho \Psi^{te} = \frac{1}{2} \varepsilon_{ij}^e E_{ijkl} \varepsilon_{kl}^e - \beta_{ij} \varepsilon_{ij}^e (T - T_r) - \rho \eta_r (T - T_r) - \frac{1}{2} \rho c_p (T - T_r)^2 - \frac{1}{2} k_{ij} \frac{\nabla_i T \nabla_j T}{\dot{T}} \quad (45)$$

where $E(\phi, \nabla^2 \phi)$ is the fourth-order damage elastic tensor given in equation (12). The thermal rate of deformation is characterised by $\beta(\phi, \nabla^2 \phi)$, which is a second-order

tensor containing the coefficients of thermal expansion. Both E and β are functions of the damage variables ϕ and $\nabla^2\phi$. T_r is the reference temperature, η_r is the reference entropy, c_p is the specific heat at constant pressure, and $k_{ij} = k\delta_{ij}$ is the heat conductivity second-order tensor (k being the conductivity coefficient and δ_{ij} is the Kronecker delta).

The thermo-viscoplastic and thermo-viscodamage energies, Ψ^{vp} and Ψ^{vd} , are assumed to have the following quadratic forms:

$$\rho\Psi^{vp} = \frac{1}{2}(a_1p^2\mathcal{G} + a_2\alpha_{ij}\alpha_{ij})\mathcal{G}; \quad \rho\Psi^{vd} = \frac{1}{2}a_3r^2\mathcal{G} \quad (46)$$

where a_k ($k=1,\dots,3$) are material-dependent constants which are considered here independent of temperature. The temperature term \mathcal{G} is the homologous temperature defined as $\mathcal{G} = 1 - [(T - T_r)/(T_m - T_r)]^n$, where T_m is the melting temperature and n is the temperature softening component, which might be assumed different for each plasticity and damage mechanisms. However, n is assumed here to be the same for both plasticity and damage.

The proposed definition of Ψ allows the derivation of the constitutive equations and the internal dissipation described next. The constitutive equations for stress, equation (30)₁, can be written from the thermodynamic potential equation (45) and equations (17) and (41) as follows:

$$\sigma_{ij} = E_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^{vp} - \varepsilon_{kl}^{vd}) - \beta_{ij}(T - T_r) \quad (47)$$

where it can be directly concluded that

$$E_{ijkl} = \rho \frac{\partial^2 \Psi}{\partial \varepsilon_{ij}^e \partial \varepsilon_{kl}^e}, \quad \beta_{ij} = -\rho \frac{\partial^2 \Psi}{\partial \varepsilon_{ij}^e \partial T} \quad (48)$$

Furthermore, by using equations (9), (12), and (11), the nominal stress relation equation (47) can be written in the effective (undamaged) configuration as follows:

$$\bar{\sigma}_{ij} = \bar{E}_{ijkl}(\bar{\varepsilon}_{kl} - \bar{\varepsilon}_{kl}^{vp}) - \bar{\beta}_{ij}(T - T_r) \quad (49)$$

such that $\bar{\beta}$ is defined as:

$$\bar{\beta}_{ij} = \hat{M}_{ijkl}\beta_{kl} \quad (50)$$

where \hat{M} is given in equation (10), and $\bar{\beta} = 3K^e\alpha_r\mathbf{1}$ with K^e is the effective bulk modulus, α_r is the coefficient of thermal expansion, and $\mathbf{1}$ is the second-order unit tensor.

The constitutive equation for entropy, equation (30)₂, can be written from the thermodynamic potential, equations (45) and (46), assuming the decoupling between the thermal effects induced through elasticity, viscoplasticity, and viscodamage such that:

$$\eta = \eta^{te} + \eta^{vp} + \eta^{vd} \quad (51)$$

where

$$\eta^{te} = \eta_r + c_p (T - T_r) + \frac{1}{\rho} \beta_{ij} (\varepsilon_{ij} - \varepsilon_{ij}^{vp} - \varepsilon_{ij}^{vd}) \quad (52)$$

$$\eta^{tvp} = \frac{1}{2\rho} [a_1 p^2 + a_2 \alpha_{ij} \alpha_{ij}] Z; \quad \eta^{tvd} = \frac{1}{2\rho} a_3 r^2 Z \quad (53)$$

In the above equations Z is given by:

$$Z = -\frac{\partial \mathcal{G}}{\partial T} = \frac{n}{T_m - T_r} \left(\frac{T - T_r}{T_m - T_r} \right)^{n-1} \quad (54)$$

The constitutive equation for the heat flux vector \mathbf{q} can be obtained from equations (30)₃ and (45) as follows:

$$q_i = -k_{ij} \nabla_j T \quad (55)$$

which is the well-known Fourier heat conduction law. The negative sign indicates the heat flow is opposite to the direction of temperature increase.

The state laws of the assumed internal state variables in equations (31) and (32) are obtained from equation (46) at constant temperature as follows:

$$R = a_1 p \mathcal{G}; \quad X_{ij} = a_2 \alpha_{ij} \mathcal{G}; \quad K = a_3 r \mathcal{G} \quad (56)$$

$$Y_{ij} = \frac{1}{2} [\sigma_{mn} - \beta_{mn} (T - T_r)] \widehat{M}_{manb} J_{arbsij} E_{rspq}^{-1} [\sigma_{pq} + \beta_{pq} (T - T_r)] \quad (57)$$

$$Y_{ij}^g = \frac{\ell^2}{4} Y_{ij} \quad (58)$$

where \mathbf{J} is a sixth-order constant tensor and is given by:

$$J_{arbsij} = -\frac{\partial \widehat{M}_{arbs}^{-1}}{\partial \phi_{ij}} = -\frac{1}{2} \ell^2 \frac{\partial \widehat{M}_{arbs}^{-1}}{\partial \nabla^2 \phi_{ij}} = -\frac{\partial \widehat{M}_{arbs}^{-1}}{\partial \phi_{ij}} = \frac{1}{2} (\delta_{ar} \delta_{bt} \delta_{sj} + \delta_{ai} \delta_{rj} \delta_{bs}) \quad (59)$$

Note that equation (10) is used in obtaining the expression in the above equation.

3.3 Viscoplasticity and viscodamage potentials

The next important step is the selection of the appropriate forms for the viscoplastic and viscodamage potentials, F and G , in order to establish the desired constitutive equations that describe the mechanical behaviour of the material. It is clearly seen in the previous part of this work that the viscodamage evolution laws are strongly coupled with viscoplasticity. To maintain this strong coupling, two independent damage mechanisms are distinguished. One mechanism is coupled with viscoplasticity, while the other occurs independent of viscoplastic deformation. The first mechanism is dominant in the case of shear stresses and the second is due to hydrostatic stresses. In order to obtain

non-associative flow rules for the flux variables p , α , and r in equations (43) and (44), one can assume the existence of F and G such that they are, respectively, not equal to viscoplasticity yield condition, $f = 0$, and the viscodamage evolution condition, $g = 0$. This postulate is essential in order to obtain nonlinear evolutions of the viscoplastic and viscodamage hardening rules, which gives a more realistic characterisation of the material response in the deformation process. Moreover, once a material is damaged, further loading can only affect the undamaged region. Thus, the damage function G is defined in terms of the effective stresses and strains. By combining viscoplasticity with damage, it seems natural that viscoplasticity can only affect the undamaged material skeleton. Therefore, the viscoplasticity function F is also defined in terms of the effective stresses and strains. Therefore, the following quadratic analytical expressions are assumed for F and G in terms of effective quantities such that

$$F = f + \frac{1}{2}k_1\bar{R}^2 + \frac{1}{2}k_2\bar{X}_{ij}\bar{X}_{ij} \quad (60)$$

$$G = g + \frac{1}{2}k_3K^2 \quad (61)$$

where k_i ($i=1-3$) are material constants used to adjust the units of the terms comprising the above equations. \bar{X} is the effective backstress tensor associated with the kinematic hardening and is expressed as follows:

$$\bar{X}_{ij} = \hat{M}_{ijkl}X_{kl} \quad (62)$$

Furthermore, the viscoplasticity isotropic hardening function \bar{R} , which is defined in the effective (undamaged) configuration, represents a global expansion in the size of the yield surface $f = 0$ with no change in shape. Thus, for a given yield criterion and flow rule, isotropic hardening in any process can be predicted from the knowledge of the function \bar{R} which, in principle, can be determined from a single test (e.g. the uniaxial test). Therefore, \bar{R} is related to the nominal isotropic hardening function R as follows:

$$\bar{R} = \frac{R}{1-r} \quad (63)$$

where r is defined as the accumulative viscodamage and expressed as follows:

$$r = \int_0^t \dot{r} dt, \quad \dot{r} = \sqrt{\hat{\phi}_{ij}\hat{\phi}_{ij}} \quad (64)$$

Note that if \bar{A} is any arbitrary variable in the effective configuration, the following equality holds true:

$$\hat{\bar{R}} = \bar{R}, \quad \hat{\bar{X}}_{ij} = \bar{X}_{ij} \quad (65)$$

This Lemma simply implies that the nonlocal value of any quantity in the effective configuration is equal to its effective nonlocal value (Voyiadjis et al., 2004; Abu Al-Rub and Voyiadjis, 2006).

It is noteworthy that replacing the local damage variable ϕ by its nonlocal counterpart $\hat{\phi}$ in equation (8) makes any state variable or function defined in the undamaged configuration a nonlocal quantity as is evident by equations (9), (62), (63), and (64). Therefore, selecting $\hat{\phi}$ as a nonlocal quantity transforms all the developed evolution laws to nonlocal ones, which makes $\hat{\phi}$ as an ideal assumption for strong enhancement of nonlocality.

The assumed potential functions, F and G , indicate the need for two loading surfaces f and g , one for viscoplasticity and another for viscodamage, respectively. Thus, the coupled anisotropic viscodamage and viscoplasticity formulation is a two-surface model whereby anisotropic viscodamage is formulated in the spirit of viscoplasticity, complete with a viscodamage criterion and flow rules. However, it is possible to activate one or both of the surfaces depending on the corresponding criteria for viscoplasticity and viscodamage.

3.4 Thermo-viscoplasticity condition

A generalised dynamic surface, f , based on the Perzyna unified viscoplasticity model (Perzyna, 1963; 1971) is utilised here (Voyiadjis et al., 2003; 2004; Abu Al-Rub and Voyiadjis, 2006):

$$f = \sqrt{3J_2} - [\bar{\sigma}_{yp} + \bar{R}][1 + (\eta^{vp} \dot{\bar{p}})^{1/m}] \mathcal{G} \leq 0 \quad (66)$$

where $J_2 = 1/2(\bar{\boldsymbol{\tau}} - \bar{\boldsymbol{X}}) : (\bar{\boldsymbol{\tau}} - \bar{\boldsymbol{X}})$ is the second invariant of the resultant deviatoric stress tensor $(\bar{\boldsymbol{\tau}} - \bar{\boldsymbol{X}})$; $\bar{\boldsymbol{\tau}}$ is the effective deviatoric part of the Cauchy stress tensor; $\bar{\sigma}_{yp}$ is the initial yield stress in the undamaged state at a reference temperature, zero viscoplastic strain, and static strain rate; η^{vp} is the viscosity or fluidity parameter, which is referred to as the relaxation time according to the notation given by Perzyna (1988); $\dot{\bar{p}}$ is the effective accumulative viscoplastic strain; m is the rate-sensitivity exponent; and $\mathcal{G} = 1 - [(T - T_r) / (T_m - T_r)]^n$ is the homologous temperature with n being the temperature softening parameter, which is different from m . The deviatoric stress $\bar{\boldsymbol{\tau}}$ can be expressed in terms of the damage effect tensor $\hat{\boldsymbol{M}}$, equation (10), as follows (Voyiadjis and Kattan, 1999):

$$\bar{\tau}_{ij} = \hat{M}'_{ijkl} \sigma_{kl} \quad \text{with} \quad \hat{M}'_{ijkl} = \hat{M}_{ijkl} - \frac{1}{3} \hat{M}_{rkrj} \delta_{ij} \quad (67)$$

The criterion in equation (66) is a generalisation of the classical rate-independent von-Mises yield criterion for rate-dependent materials. The latter can be simply recovered by imposing $\eta^{vp} = 0$ (no rate effect), so that one has the case $f \leq 0$. It also allows a generalisation of the standard Kuhn-Tucker loading/unloading conditions (Voyiadjis et al., 2003; 2004):

$$f \leq 0, \quad \dot{\lambda}^{vp} \geq 0, \quad \dot{\lambda}^{vp} f = 0, \quad \dot{\lambda}^{vp} \dot{f} \equiv 0 \quad (68)$$

In equation (66), f can expand and shrink not only by softening/hardening effects, but also due to softening/hardening rate effects. The right-hand-side of equation (66) defines the flow stress as a function of strain, strain-rate, and temperature and it converges to a great extent to the classical (local) constitutive laws of Johnson and Cook (1985), Zerilli and Armstrong (1987), and Voyiadjis and Abed (2005).

The effective rate of the accumulative viscoplastic strain, $\dot{\bar{p}}$, is defined by:

$$\dot{\bar{p}} = \sqrt{\frac{2}{3} \dot{\bar{\epsilon}}_{ij}^{vp} \dot{\bar{\epsilon}}_{ij}^{vp}} \quad (69)$$

where $\dot{\bar{\epsilon}}^{vp}$ is the viscoplastic strain rate in the effective configuration. Moreover, one can adopt the energy-correlating hypothesis; each type of energy process in the damaged state is equal to the corresponding state in the effective configuration. Thus, $\dot{\bar{\epsilon}}^{vp}$ can be related to $\dot{\epsilon}^{vp}$, equation (42)₁, through the use of the viscoplastic strain energy equivalence hypothesis (Lee et al., 1985; Voyiadjis and Deliktas, 2000), which states that the viscoplastic energy in terms of the effective and nominal stress and strain quantities must be equal. This gives the following relation, such that:

$$\dot{\bar{\epsilon}}_{ij}^{vp} = \widehat{M}_{ijkl}^{-1} \dot{\epsilon}_{kl}^{vi} \quad (70)$$

where \widehat{M}^{-1} is the inverse of the fourth-order damage tensor given by equation (13).

Substituting equation (42)₁ into equation (70) and making use of equation (9), and the chain rule, yields:

$$\dot{\bar{\epsilon}}_{ij}^{vp} = \dot{\lambda}^{vp} \frac{\partial f}{\partial \bar{\sigma}_{ij}} \quad (71)$$

Since $\partial f / \partial \bar{\sigma} : \partial f / \partial \bar{\sigma} = 1.5$, it can be easily shown that $\dot{\bar{p}}$ defined by equation (69) is related to $\dot{\lambda}^{vp}$ by:

$$\dot{\bar{p}} = \dot{\lambda}^{vp} \quad (72)$$

Substituting equation (60) into equation (43)₁ and making use of the chain rule with equation (63), and finally comparing the result with equation (72) yields the following expression:

$$\dot{p} = \frac{\dot{\bar{p}}}{1-r} (1 - k_1 \bar{R}) \quad (73)$$

Note that in obtaining the above equation, $\partial f / \partial R = -1$ is used since the evolution of R characterises the radius of the yield surface in the absence of the strain rate and temperature effects.

By taking the time derivative of equations (56)₁ and (63) for a given temperature while maintaining the damage history constant (i.e. the damage internal state variable ϕ and the temperature T are kept constant) and substituting equation (73), one can write the following evolution equation for \bar{R} as follows:

$$\dot{\bar{R}} = \frac{a_1 \mathcal{G}}{(1-r)^2} (1 - k_1 \bar{R}) \dot{\bar{p}} \quad (74)$$

In order to derive the kinematic hardening evolution equation, one first makes use of the $\dot{\alpha}$ law in equation (43)₂ along with the chain rule and equation (62), such that one can write the following relation:

$$\dot{\alpha}_{ij} = -\dot{\lambda}^{vp} \widehat{M}_{minj} \frac{\partial F}{\partial \bar{X}_{mn}} \quad (75)$$

Substituting equations (60) and (72) into the above equation yields:

$$\dot{\alpha}_{ij} = -\dot{\bar{p}} \widehat{M}_{minj} \left(\frac{\partial f}{\partial \bar{X}_{mn}} + k_2 \bar{X}_{mn} \right) \quad (76)$$

Since $\partial f / \partial \bar{X} = -\partial f / \partial \bar{\epsilon} = -\partial f / \partial \bar{\sigma}$, as it is clear from equation (66), it can be easily shown by using equations (71) and (72) that equation (76) can be rewritten as follows:

$$\dot{\alpha}_{ij} = \widehat{M}_{minj} \left(\dot{\bar{\epsilon}}_{mn}^{vp} - k_2 \bar{X}_{mn} \dot{\bar{p}} \right) \quad (77)$$

However, taking the time derivative of equations (56)₂ and (62) for a given temperature and keeping the damage history constant (i.e. the viscodamage effective tensor \widehat{M} and the temperature T are kept constant), one can write the following evolution equation for \bar{X} :

$$\dot{\bar{X}}_{ij} = \widehat{M}_{ikjl} \widehat{M}_{mkl} \left(a_2 \dot{\bar{\epsilon}}_{mn}^{vp} - k_2 a_2 \dot{\bar{p}} \bar{X}_{mn} \right) \mathcal{G} \quad (78)$$

When the rate and temperature independent response of a material is assumed and the micro-damage effect is neglected, then the kinematic hardening law given by equation (78) reduces to that proposed by Armstrong and Frederick (1966).

It is noteworthy that the derived evolution equations for \bar{R} , equation (74), and \bar{X} , equation (78), contain hardening terms that represent the strengthening mechanism and recovery terms that represent the softening mechanism. Both the hardening and recovery terms are affected by the static (thermal) recovery term \mathcal{G} in such a way that the functional dependence of the hardening of materials could be multiplicatively decomposed into two functions, thermal (static) and stress (dynamic). Therefore, those evolution equations characterise the time and thermal effects due to the rate and temperature dependency.

Moreover, it is worth mentioning that the formulated thermo-viscoplasticity constitutive equations can be modified to include static (annealing) and strain-driven (creep) microstructural grain and dispersed particle coarsening effect due to creep and very aggressive thermo-mechanical cycling during the service life of a microelectronic solder joint with homologous temperatures reaching $\mathcal{G} = 0.9$. This is a serious problem for solder joint reliability (see for example Tang and Basaran, 2001; Basaran and Wen, 2003; Dutta et al., 2005). It follows that the yield stress in equation (66), $[\bar{\sigma}_{yp} + \bar{R}][1 + (\eta^{vp} \dot{\bar{p}})^{1/m}] \mathcal{G}$, decreases with coarsening of the grain size and particle size

such that the creep rate for solder alloys highly depends on the grain size and particle size. Specifically, one can express $\bar{\sigma}_{yp}$ and η^{vp} in terms of the grain and particle sizes such that the size evolves as a function of the creep strain, temperature, and time. The experimental and theoretical works by, for example, Tang and Basaran (2001), Basaran and Wen (2003), Dutta et al. (2005), and the references cited therein on coarsening in solder materials can be used to guide this development.

3.5 Thermo-viscodamage condition

The anisotropic viscodamage governing equations are viscoplasticity-like. Analogous to the viscoplastic surface presented in the previous subsection, a viscodamage surface g can then be simply assumed as follows (Voyiadjis et al., 2003; 2004; Abu Al-Rub and Voyiadjis, 2006):

$$g = \sqrt{(Y_{ij} + Y_{ij}^g)(Y_{ij} + Y_{ij}^g)} - [K_o + K][1 + (\eta^{vp} \dot{p})^{1/m}] \mathcal{G} \leq 0 \quad (79)$$

where the damage driving forces \mathbf{Y} and \mathbf{Y}^g are given by equations (57) and (58), respectively; and K_o is the damage threshold at reference temperature and static strain rate, which characterises the incipient of damage nucleation.

The model response in the viscodamage domain is then characterised by the Kuhn-Tucker complementary conditions as follows:

$$g \leq 0, \quad \dot{\lambda}^{vd} \geq 0, \quad \dot{\lambda}^{vd} g = 0, \quad \dot{\lambda}^{vd} \dot{g} = 0 \quad (80)$$

In order to derive the viscodamage isotropic hardening evolution equation, one follows the same procedure presented in the former section for viscoplasticity. By substituting equation (61) into equation (44)₁ and then substituting the result into the time derivative of equation (56)₃, the following relation is obtained:

$$\dot{K} = a_3 (1 - h_1 \widehat{K}) \dot{\lambda}^{vd} \mathcal{G} \quad (81)$$

Similar to the viscoplastic hardening evolution equations, the derived viscodamage evolution equation considers the dynamic recovery as well as the static recovery (thermal recovery). Besides, the static recovery occurs in both the hardening and the dynamic recovery terms. The hardening term in each of the assumed internal state variable accounts for strengthening mechanisms, while the recovery term accounts for softening mechanisms.

4 Rate-type constitutive relation

Operating on the stress-strain relation in equation (47) with the time derivative yields:

$$\dot{\sigma}_{ij} = E_{ijkl} (\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^{vp} - \dot{\epsilon}_{kl}^{vd}) + \dot{E}_{ijkl} \epsilon_{kl}^e - \beta_{ij} \dot{T} - \dot{\beta}_{ij} (T - T_r) \quad (82)$$

where \dot{E} and $\dot{\beta}$ are given by:

$$\dot{\mathbf{E}}_{ijkl} = \frac{\partial E_{ijkl}}{\partial \widehat{\mathbf{M}}_{abuv}^{-1}} \dot{\widehat{\mathbf{M}}}_{abuv}^{-1} = \frac{\partial E_{ijkl}}{\partial \widehat{\mathbf{M}}_{abuv}^{-1}} \frac{\partial \widehat{\mathbf{M}}_{abuv}^{-1}}{\partial \widehat{\phi}_{mn}} \dot{\widehat{\phi}}_{mn} \quad (83)$$

$$\dot{\beta}_{ij} = \frac{\partial \beta_{ij}}{\partial \widehat{\mathbf{M}}_{abuv}^{-1}} \dot{\widehat{\mathbf{M}}}_{abuv}^{-1} = \frac{\partial \beta_{ij}}{\partial \widehat{\mathbf{M}}_{abuv}^{-1}} \frac{\partial \widehat{\mathbf{M}}_{abuv}^{-1}}{\partial \widehat{\phi}_{mn}} \dot{\widehat{\phi}}_{mn} \quad (84)$$

In the effective configuration, the rate of effective stress tensor can be written from equation (49) as follows:

$$\dot{\bar{\sigma}}_{ij} = \bar{E}_{ijkl} \left(\dot{\bar{\epsilon}}_{kl} - \dot{\bar{\epsilon}}_{kl}^{vp} \right) - \bar{\beta}_{ij} \dot{T} \quad (85)$$

Making use of equations (12), (17), (49), (50), and (59), one can then rewrite equation (82), after some lengthy manipulations, as follows:

$$\dot{\sigma}_{ij} = E_{ijkl} \left(\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^{vi} \right) - A_{ijkl} \dot{\phi}_{kl} - \beta_{ij} \dot{T} \quad (86)$$

where

$$A_{ijkl} = \left[-\frac{\partial E_{ijmn}}{\partial \widehat{\mathbf{M}}_{abuv}^{-1}} \epsilon_{mn}^e + \frac{\partial \beta_{ij}}{\partial \widehat{\mathbf{M}}_{abuv}^{-1}} (T - T_r) \right] \frac{\partial \widehat{\mathbf{M}}_{abuv}^{-1}}{\partial \widehat{\phi}_{kl}} = J_{ibjvkl} \bar{\sigma}_{bv} + \widehat{\mathbf{M}}_{ipjq}^{-1} \bar{E}_{pqau} \epsilon_{bv}^e J_{abuvkl} \quad (87)$$

It is noteworthy that the above incremental stress-strain relation can be found similarly using the stress-strain relation in the effective configuration in equation (85). This equation shows that softening occurs due to the thermal and damage evolution.

5 Thermo-mechanical coupling

Besides the thermal loading of microelectronics solder alloys, a local increase in temperature may influence the material behaviour during deformation. Substituting the internal energy density, e , from equation (25) into the first law of thermodynamics, equation (23), yields the following energy balance equation:

$$\rho \left(\dot{\Psi} + \eta \dot{T} + \dot{\eta} T \right) - \sigma_{ij} \dot{\epsilon}_{ij} - \rho r_{ext} + q_{i,i} = 0 \quad (88)$$

Substituting $\dot{\Psi}$ from equations (27) and (28) into the above equality and taking into account the results in equations (30), (34), and (55) yields the following:

$$\rho \dot{\eta} T = \sigma_{ij} \left(\dot{\epsilon}_{ij}^{vp} + \dot{\epsilon}_{ij}^{vd} \right) - \Pi_{int} + \rho r_{ext} + k_{ij} \nabla_i T \nabla_j T + \frac{1}{T} k_{ij} \nabla_i \dot{T} \nabla_j T \quad (89)$$

where Π_{int} is given in equation (34). Furthermore, according to the definition given by equations (51)–(53) for the specific entropy $\eta = \tilde{\eta}(\boldsymbol{\epsilon}^e, T, \mathfrak{N}_k)$ ($k = 1, \dots, 5$); operating on the entropy relation, equation (30)₂, with the time derivative and substituting the result into equation (89) gives the thermo-mechanical heat balance equation as follows:

$$\rho c_p \dot{T} = \sigma_{ij} \left(\dot{\varepsilon}_{ij}^{vp} + \dot{\varepsilon}_{ij}^{vd} \right) + \sum_{k=1}^5 \left(\frac{\partial \Sigma_k}{\partial T} T - \Sigma_k \right) \dot{\Sigma}_k - \beta_{ij} \dot{\varepsilon}_{ij}^e T + \rho r_{ext} + k_{ij} \nabla_i T \nabla_j T \quad (90)$$

where $c_p = T \partial \eta / \partial T$ is the tangent specific heat capacity at constant pressure used to approximate the specific heat capacity at constant stress.

The left-hand side of equation (90) represents the total thermal dissipation. The first term on the right-hand side represents the rate of viscoelastic work converted to heat; the second term represents the rate of internal heat generation that encompasses all dissipative processes excluding the gross viscoelastic deformation that is present during viscoplastic and viscodamage deformation; the third term is the reversible thermo-elastic effect; the fourth term represents an external heat source; and the last term is due to the parabolic heat conduction. Moreover, solder joints are particularly prone to failure during a drop. If the duration of the event is sufficiently short as when dropping a cell phone or a laptop, the thermal conduction term becomes negligible, i.e. adiabatic conditions prevail. Therefore, the damage behaviour of solders at high strain rates could be an important design parameter.

Furthermore, Abu Al-Rub and Voyiadjis (2006) showed that equation (90) can be approximated by:

$$\rho c_p \dot{T} = \Upsilon \sigma_{ij} \left(\dot{\varepsilon}_{ij}^{vp} + \dot{\varepsilon}_{ij}^{vd} \right) - \beta_{ij} \dot{\varepsilon}_{ij}^e T + \rho r_{ext} + k \nabla^2 T \quad (91)$$

where Υ is the fraction of the viscoelastic work rate converted into heat which implicitly takes into account the second term in the left-hand-side of equation (90) and $k_{ij} = k \delta_{ij}$ is also substituted.

The rate type-equations, equation (86) and (91), take into account the effects of viscoplasticity and viscodamage strain-induced anisotropy (i.e. kinematic hardening), flow stress (i.e. isotropic hardening), temperature and strain-rate sensitivity, anisotropic damage (i.e. softening generated by damage defects nucleation and growth mechanisms), thermo-mechanical coupling (i.e. thermal viscoplastic softening and thermal expansion), strong viscoplasticity and viscodamage coupling, and heterogeneity in the material behaviour (i.e. the nonlocal influence). These features make the proposed model more general such that it can be used for modeling solder alloys as well as other heterogeneous systems.

In a future work, the developed theory will be implemented in a finite element code using the numerical approach developed by Abu Al-Rub and Voyiadjis (2005) for gradient-dependent models. Then numerical examples will be presented to show the effectiveness of the proposed model in accurately predicting the various microstructural and deformation aspects of solder alloys under different loading conditions.

6 Physical interpretation of the material length scale

The full utility of the proposed gradient-type theory hinges on one's ability to determine the constitutive length scale parameter ℓ that scales the gradient effect. Therefore, ℓ must be calibrated from micromechanical tests where size effects and plasticity and damage heterogeneities are encountered [see Abu Al-Rub and Voyiadjis (2004a; 2004b) and Abu Al-Rub (2007) for more details]. Moreover, there have been some attempts to

interpret the physical nature of ℓ based on microstructural arguments. For example, based on the Taylor dislocation model, Nix and Gao (1998) have identified ℓ as the square of average dislocation spacing, λ , over the Burgers vector, b (i.e. $\ell = \lambda^2/b$). In terms of macroscopic quantities, $\ell \propto b(\mu/\sigma_{yp})^2$, where μ is the shear-modulus and σ_{yp} is the yield strength from a uniaxial test. Abu Al-Rub and Voyiadjis (2004a; 2004b) and Abu Al-Rub (2007) have developed a micromechanical model that nonlinearly couples SSDs and GNDs (Ashby, 1970) and used it to derive an analytical form for ℓ in terms of measurable microstructural physical parameters, where it was shown that ℓ is proportional to the mean free-path distance between dislocations (i.e. $\ell = A\lambda$ with A being a material constant and λ evolves with the course of inelastic deformation) and it can be best identified from micro- and nano-indentation experiments. Gomez and Basaran (2005; 2007) have used successfully the indentation size effect model by Abu Al-Rub and Voyiadjis (2004a) for the identification of the material length scale of microelectronic solder alloys.

Moreover, many authors tend to use a constant value for the length scale parameter and neglecting its evolution with the state of deformation. For example, the damage zone ahead of the crack tip or the mean dislocation spacing will decrease with increasing strain rate and increase with decreasing temperature, which for small scale yielding is of the order of microns. This causes the intrinsic material length scale to decrease with increasing strain rates and to increase with temperature decrease. However, opposite behaviour is anticipated for the gradient term; that is, gradients are inversely proportional to the length scale over which plastic and/or damage deformations occur. Therefore, taking into account the strain rate and temperature effects is crucial to the evolution of the length scale in solder alloys. Motivated by this crucial observation, Abu Al-Rub and Voyiadjis (2004b) proposed the following evolution equation for ℓ :

$$\dot{\ell} = \ell \nu_o \exp \left[- (U_o/kT) \left\{ 1 - (\sigma^*/\sigma_o^*)^p \right\}^q \right] \quad (92)$$

with $\ell = 18\alpha^2 (\mu/\sigma_{yp})^2 b$, where α is an empirical parameter (ranges from 0.1–0.3); ν_o is the fundamental vibrational frequency of the dislocation, p and q are material constants, k is the Boltzmann's constant, U_o is the referential activation energy, σ^* is the thermal stress, and σ_o^* is the reference thermal stress.

Therefore, ℓ is one of the most important material parameters to be identified and for the proposed model to yield improved quantitative results. The identification procedures for the other material parameters associated with the proposed model have been discussed by the author and others in several publications. There are 24 material constants that need to be determined. The material constants σ_{yp} , μ , K^e , ρ , α_i , c_p , k , and b are standard material constants and can be easily obtained from the literature for any material. The identification procedure for the material constants associated with isotropic hardening, a_1 and k_1 , and kinematic hardening, a_2 and k_2 , can be found in Voyiadjis and Abu Al-Rub (2003). For the material constants associated with damage evolution, K_o , a_3 , and h_1 , one can consult Abu Al-Rub and Voyiadjis (2003). The material constants associated with rate and temperature sensitivity, n , η^{vp} , and m , the reader is

referred to Voyiadjis et al. (2006). Detailed discussion about the material constant α associated with the initial value of ℓ has been presented by Abu Al-Rub (2007). Finally, the material constants associated with the evolution of the material length scale in equation (92), ν_o , k , U_o , p , and q , for different materials can be found in Voyiadjis and Abed (2005) and the references quoted therein.

Finally, models that are based on fundamental physical laws that govern dislocation motion, micro-cracks propagation, and voids growth, nucleation, and coalescence and their interaction with various defects and interfaces could be used to shed some insights on how the evolution of the damage and plasticity length scales in solder alloys may look like. Moreover, for polycrystalline lead-free solders, the proposed constitutive model may also be effective, as they commonly consist of heterogeneous microstructure with fine scale precipitates and sometimes very large intermetallic inclusions. Computational studies using the proposed model for simulating the above arguments will be presented in future works by the author.

7 Summary

In the current paper, the systematic construction of a thermodynamic consistent gradient-enhanced framework for heterogeneous solder materials used in microelectronic packages, which provides a strong coupling between viscoplasticity and anisotropic viscodamage, is presented. Thermodynamic consistency is restored through the derived constitutive equations in order to introduce and address issues such as the statistical inhomogeneity in the evolution related viscoelasticity macroscopic variables, and temperature and strain-rate sensitivity. This framework is general enough to describe the evolution of viscoelasticity in a material body accounting for physical discontinuities through the use of a nonlocal approach based on the gradient-dependent theory. The interaction of the length scales is a crucial factor in understanding and controlling the distribution and size of the material micro-defects influence on the macroscopic response. The behaviour of these defects is captured not only individually, but also the enhanced strong coupling between the two dissipative processes takes into account the interaction between these defects and their ability to create spatial-temporal patterns under thermo-mechanical loading conditions.

Length scale parameters are implicitly and explicitly introduced into the present dynamical formalism. Implicit length scale measure is introduced through the use of the rate-dependent theory, while an explicit length scale parameter is introduced through the use of the gradient-dependent damage theory. Moreover, a strong coupling between the two dissipative processes, viscoplasticity and viscodamage, is implemented. This strong coupling is assessed by using two separate viscoplasticity and viscodamage surfaces with coupled non-associated flow rules in such a way that both viscoplastic and viscodamage rules are dependent on the viscoplastic and viscodamage potentials. Two viscodamage mechanisms have been considered, one mechanism is coupled with viscoplasticity, and while the other occurs independent of viscoplastic deformation. The dissipation function of the latter occurs in both the elastic and viscoplastic domains. The constitutive equations for the damaged material are written according to the principle of strain energy equivalence between the virgin material and the damaged material.

One needs more studies to be performed in order to effectively assess the potential applications for the proposed theoretical framework. Additional aspects for future studies is the consideration of the dependency of the intrinsic material length scale on the strain-rate and temperature variation, testing the viscoplasticity and viscodamage consistency conditions, and detailed description of solving the present coupled system of equations.

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